THE APPLICATION OF MODERN THEORIES OF INTEGRATION TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

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1. **Introduction.** It is the purpose of this paper to present a method of applying the modern theories of divergent and Stieltjes' integrals to the discussion of certain integrals closely related to the Fourier identity; and to present an application of this method to the solution of linear differential equations. The investigation results in assigning a meaning to a wide class of integrals which have heretofore had none; and in justifying the use of the common operations in dealing with these integrals.

In an attempt to obtain the solution of a class of electrical problems the writer was led, largely through physical arguments, to adopt formal operational methods of manipulation for the purpose of obtaining tentative results. As the work proceeded he was impressed with the large number of cases in which these methods seemed to yield correct answers, although so far as he knew they had no mathematical justification. In an attempt to remove the uncertainty in his own mind as to the breadth of the class of problems in which these methods might be used with a reasonable certainty of accurate results, he was led to formulate the argument which is presented in the following pages.

This being the origin of the work, it will not be at all surprising if it has influenced the form in which the presentation is cast. In fact, the paper is to a certain extent a companion to a technical article on "The Solution of Circuit Problems," which recently appeared in the Physical Review. Bearing in mind the fact that readers of that article may wish to refer to this, an attempt has been made to use as simple lines of argument as are consistent with rigor. This attempt has seemed to require more restrictive conditions upon some of the preliminary theorems than would otherwise have been necessary; but has left them in all cases sufficiently broad for the purposes of this discussion.

Broadly speaking, there are three main divisions of the argument. In the first, which comprises sections 2 to 7, certain concepts which result from the application of the Cæsaro definition of a divergent limit to integrals are presented. In the second division, which comprises sections 8 to 10, some observations are made regarding Stieltjes' integrals which depend upon an arbitrary parameter. In the third broad division, comprising the remainder of the paper, these two discussions, which have been

carried on with very little reference to one another, are both merged in the discussion of a type of integral closely resembling the Fourier integral. This is assigned a meaning, and applied to the solution of linear differential equations.

It is perhaps desirable to add a word in acknowledgment of the sources of information which have influenced the development of the argument. Foremost among these have been* Borel's treatise on Divergent Series, a few paragraphs from Stieltjes' original work on Continued Fractions, and Hildebrandt's excellent review of the Modern Theories of Generalized Integrals. Other sources have been consulted at various times, but have exerted much less influence.

2. **The Cæsaro value for a divergent limit.** The functions to which attention will be directed in the following pages are frequently expressed in the form of limits which do not exist in the ordinary sense. It is necessary, therefore, to assign values to these limits by definition. This is best done by finding a transformation I, possessing the property that Lim $If(u)$ exists under circumstances which render Lim $f(u)$ meaningless, and also the property that Lim $If(u) \equiv$ Lim $f(u)$ whenever the latter limit exists. If such a transformation may be found, the identity Lim $If(u) \equiv$ Lim $f(u)$ may be used as a definition of its right-hand member.

It has been shown that† the operator

$$I = \frac{1}{n}\int_0^n dn \tag{1}$$

possesses the necessary properties, when n is to approach ∞. That is,

$$\operatorname*{Lim}_{n=\infty}\frac{1}{n}\int_0^n f(n)dn = \operatorname*{Lim}_{n=\infty} f(n), \tag{2}$$

whenever the limit of $f(n)$ exists; while the right-hand side of (2) fails to have a meaning for many functions for which the left-hand side exists. Furthermore, the statements

$$I(u+v) = I(u) + I(v), \qquad I(cu) = cI(u), \qquad I(c) = c \tag{3}$$

are almost immediately obvious.

For the purposes of this paper it is desirable to limit consideration to functions of the type,

$$f(n) = \int_0^n e^{inx}\phi(n)dn, \tag{4}$$

* Borel, "Lecons sur les Series Divergentes."

Stieltjes, "Sur les Fractions Continues," Annales de la Faculte' des Sciences de Toulouse, 1894–5.

Hildebrandt, "On Integrals," Bull. Amer. Math. Soc., 1917–18.

† Borel, loc. cit., p. 87.

Silverman, Transactions of Am. Math. Soc., Apr., 1916.

which are peculiar in that their Cæsaro limits can be evaluated by means of the theory of residues.

3. **Expressions of the type**

$$f(n) = \int_0^n n^j e^{inx} dn. \tag{5}$$

If j is integral the operator I may be applied to expressions such as (5). The transformed functions are then obtained by the use of the ordinary formulæ of integral calculus. Direct integration of (5) leads to an expression

$$f(n) = \sum_{k=0}^{j} a_k n^k e^{inx} - a_0, \tag{6}$$

and therefore to

$$If(n) = \sum_{k=0}^{j} a_k I n^k e^{inx} - a_0. \tag{7}$$

The terms of this series are all of the form $In^k e^{inx}$, which can be evaluated by immediate integration. The result is

$$In^k e^{inx} = \sum_{k'=1}^{k} a_{k'} n^{k'-1} e^{inx} + a_{0'} \frac{e^{inx} - 1}{n}, \tag{8}$$

where the constants depend not only on k' but also on k. It is the form of this expression, however, rather than the actual values of its coefficients, which is important.

If (8) is substituted in (7) a new expression results which may be thrown into the form

$$If(n) = \sum_{k=1}^{j} a_k' n^{k-1} e^{inx} + a_0' \frac{e^{inx} - 1}{n} - a_0.$$

Repeated application of I leads at last to

$$I^{j+1} f(n) = \sum_{k=1}^{j+1} a_0^{(k)} I^{j-k+1} \frac{e^{inx} - 1}{n} - a_0,$$

whence, by passing to the limit as $n = \infty$,

$$\operatorname*{Lim}_{n \doteq \infty} I^{j+1} f(n) = - a_0,$$

since

$$\operatorname*{Lim}_{n \doteq \infty} I^{j-k+1} \frac{e^{inx} - 1}{n} = \operatorname*{Lim}_{n \doteq \infty} \frac{e^{inx} - 1}{n} = 0.$$

The meaning of the integral $\int_0^\infty n^j e^{inx} dn$ is therefore known as soon as the value of a_0 has been determined. This leads at once to the result

$$\int_0^\infty n^j e^{inx} dn = (-1)^{j+1} \frac{\underline{|j}}{(ix)^{j+1}}. \tag{9}$$

This integral was evaluated along the real axis. It will be found illuminating to obtain its value when taken along certain other paths of integration. Suppose, for instance, it is evaluated along a line which makes an angle θ with the real axis, as shown in Fig. 1. To be explicit,

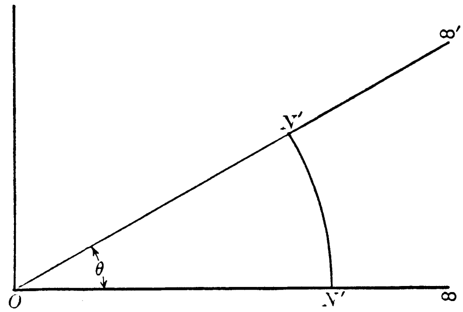

FIG. 1.

it is assumed that $|\theta| \leqq \pi/2$ and that* $\operatorname{sgn} \theta = \operatorname{sgn} x$. Along this path the integral is convergent since $\operatorname*{Lim}_{n \doteq \infty} e^{inx} = 0$. Evaluating (6) it is found that

$$\operatorname*{Lim}_{n \doteq \infty'} f(n) = -a_0,$$

which is the same as (9). This equality of the values of the integral (5) when taken along the two paths $O\infty$ and $O\infty'$ will be used in several different forms in the succeeding sections. In the notation used in Fig. 1 they may be symbolically expressed as

$$\begin{gathered} \int_{\infty' 0 \infty} = 0, \qquad \int_{\infty' N' N \infty} = 0, \qquad \int_{0 N N' \infty'} = \int_{0 \infty}, \\ \int_{N N'} = -\int_{N' O N}, \qquad \operatorname*{Lim}_{N = |N'| \doteq \infty} \int_{N N'} = 0. \end{gathered} \tag{10}$$

This argument has been carried out on the assumption that j is a positive integer. There is, however, no essential difficulty introduced when j is not integral, provided it is greater than -1. The course of the argument is unaltered except that $-a_0$ is replaced by an ordinary convergent improper integral. The value is easily determined to be $(-1)^{j+1}\Gamma(j+1)/(ix)^{j+1}$. Formula (10) also applies to this more general case.

4. **Expressions of the Type** (4). The argument concerning the general equation (4) can be made to depend upon the results obtained in section 3, provided the function $\phi(n)$ has no essential singularity at infinity. There

* The figure is drawn for $\operatorname{sgn} x = 1$.

may be essential singularities in the finite portion of the plane, but if so, it will be assumed that they are so situated that it is possible to connect them by cuts which do not anywhere intersect the real axis.* If $\phi(n)$ has at infinity a pole of order ν, it may be expanded in the series $\phi(n) = \sum_{j=-\nu}^{\infty} a_j/n^j$, which converges for all values of n satisfying the condition $|n| \geqq N$. Furthermore, the series formed from the terms $(a_j/n^j)e^{inx}$ also converges in the same region uniformly and absolutely if the imaginary part of nx is not negative. This establishes the propriety of term by term integration; hence

$$\int_N^n \phi(n)e^{inx}dn = \sum_{j=-\nu}^{\infty} a_j \int_N^n \frac{e^{inx}}{n^j} dn.$$

Now introduce the notation

$$\phi(N, n) = \sum_{j=1}^{\infty} a_j \int_N^n \frac{e^{inx}}{n^j} dn.$$

Then $\operatorname*{Lim}_{n \doteq \infty} \Phi(N, n)$ converges to the value $\Phi(N, \infty)$ in the ordinary sense of convergence. In terms of this notation (5) may be expressed as

$$\begin{aligned}\int_N^\infty \phi(n)e^{inx}dn &= \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1} \int_N^n \phi(n)e^{inx}dn \\ &= \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1}\Phi(N, n) + \sum_{j=-\nu}^{0} a_j \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1} \int_N^n \frac{e^{inx}}{n^j} dn \qquad (11) \\ &= \Phi(N, \infty) - \sum_{j=-\nu}^{0} a_j \left[\int_0^N \frac{e^{inx}}{n^j} dn + \frac{\lfloor -j}{(ix)^{-j}} \right].\end{aligned}$$

Since this equation gives the value of the limit (11) it also asserts its existence. It is possible, however, to simplify its calculation by establishing its equivalence with a certain Cauchy integral. This will be the object of the next section.

5. **Reduction to a closed path: Cauchy integral.** The integration in equation (11) was performed along the real axis. If it had been carried out along the line $ON'\infty'$ of Fig. 1 there would have been no need of applying the operator I. The result of the integration along this line could have been determined immediately to be

$$\int_{N'}^{\infty'} \phi(n)e^{inx}dn = \Phi(N', \infty') - \sum_{j=-\nu}^{0} a_j \left[\int_0^{N'} \frac{e^{inx}}{n^j} dn + \frac{\lfloor -j}{(ix)^{-j}} \right].$$

* We are aiming at an evaluation of

$$\int \phi(n)e^{inx}dn \qquad (a)$$

taken along the real axis and the condition is stated for this particular case. If we desired to carry out the integration of (a) along some path C it is only necessary to require that no cuts joining the essential singularities of $\phi(n)$ shall intersect C at any point, finite or infinite.

Hence

$$\int_{\infty' N' N \infty} \phi(n) e^{inx} dn = \Phi(N, \infty) - \Phi(N', \infty') - \sum_{j=1}^{\infty} a_j \int_{N}^{N'} \frac{e^{inx}}{n^j} dn. \quad (12)$$

The value of the left-hand side of (12) is independent of the magnitude of $N = |N'|$ so long as N is sufficiently large. The same is therefore true of the right-hand side also. Call this value L. Then as N is indefinitely increased the limit approached by both members of (12) must be this value L to which they are constantly equal. But it is almost immediately obvious that the limits of $\Phi(N, \infty)$ and $\Phi(N', \infty')$ are zero since $\operatorname*{Lim}_{n \doteq \infty} \Phi(N, n)$ converges in the ordinary sense. As for the summation, it is seen that excluding the term $j = 1$,

$$\left| \sum_{j=2}^{\infty} a_j \int_{N}^{N'} \frac{e^{inx}}{n^j} dn \right| \leqq \sum_{j=2}^{\infty} \frac{|a_j|}{N^{j-1}} \left| \theta \right| ,$$

which also approaches the limit zero. This leaves for consideration only the term $j = 1$. However, since $|\theta| \leqq \pi/2$,

$$a_1 \int_{N}^{N'} e^{inx} \frac{dn}{n} \leqq |a_1| \left| \int_0^{\theta} e^{-xN\theta/2} d\theta \leqq \frac{2|a_1|}{Nx} \right. ,$$

which also approaches the limit zero. That is, $L = 0$ and hence

$$\int_{\infty' N' N \infty} \phi(n) e^{inx} dn = 0, \qquad \int_{0 N N' \infty'} \phi(n) e^{inx} dn = \int_{0 \infty} \phi(n) e^{inx} dn. \quad (13)$$

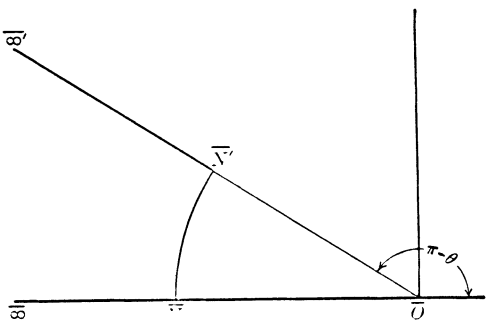

FIG. 2.

It is quite obvious that a similar argument may be applied to the integral

$$\int_{-\infty}^{0} \phi(n) e^{inx} dn.$$

In the notation of Fig. 2 the results of such an argument are

$$\int_{\overline{\infty}' \overline{N}' \overline{N} \overline{\infty}} \phi(n) e^{inx} dn = 0, \qquad \int_{\overline{\infty}' \overline{N}' \overline{N} \overline{0}} \phi(n) e^{inx} dn = \int_{\overline{\infty} \overline{0}} \phi(n) e^{inx} dn. \quad (14)$$

Now let $\theta = \pi/2$ in both (13) and (14); then $\overline{\infty}' = \infty'$ and $\overline{N}' = N'$ so that in Figs. (1) and (2) the paths $\infty'N'$ and $\overline{\infty}'\overline{N}'$ either coincide or lie vertically above one another on different sheets of the Riemann surface for $\phi(n)$. The conditions which have been imposed upon the singularities of ϕ are sufficient, however, to justify the statement that if the points O and $\overline{O}$ lie upon the same sheet of this Riemann surface the points N' and N' will also lie upon the same sheet, so that the integrals along the paths $\overline{\infty}'\overline{N}'$ and $\infty'N'$ are equal. This results finally in the equation

$$\int_{-\infty}^{\infty} \phi(n)e^{inx}dn = \int_{N'\overline{N}0NN'} \phi(n)e^{inx}dn. \tag{15}$$

That is, in words:

THEOREM 1. *The Caesaro value of the integral*

$$\int_{-\infty}^{\infty} \phi(n)e^{inx}dn$$

taken along the real axis is identical with the integral of the same function taken about the path shown in Fig. 3, *provided*: (*a*) *the function* $\phi(n)$ *is essentially singular at infinity;* (*b*) *the essential singularities of* ϕ *can be joined by a set of cuts which nowhere intersect the path of integration; and* (*c*) *the radius* N *of the circular part of the path is larger than the modulus of* n *for that singular point of* ϕ *which is farthest removed from the origin.*

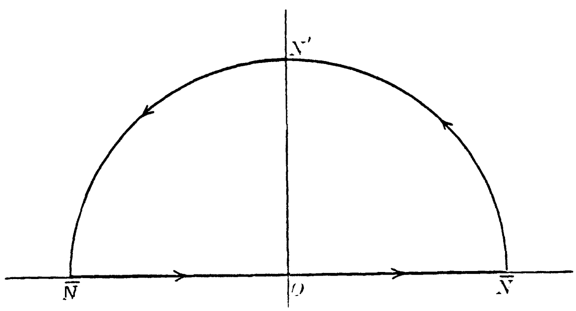

FIG. 3.

Of course, if $x < 0$, the point N' must be taken below instead of above the real axis, while if $x = 0$ the argument fails altogether.

6. **Differentiation and integration under the sign of integration.** In every case except $x = 0$ the divergent integral (4) has been evaluated in the Cæsaro sense, and the result has been proved to be identical with a Cauchy integral in the complex plane. This opens up the possibility of applying much of the elementary theory of analytical functions to the particular type of divergent integrals with which this paper is concerned.

It is not intended, however, to follow out this line of development, except to the extent of noting a few of those facts regarding differentiation and integration under the integral sign which will be needed in the work which follows.

The possibility of expanding $\phi(n)$ in an absolutely and uniformly convergent series of the form $\phi(n) = \sum_{j=-\nu}^{\infty} a_j/n^j$ has been assumed. Consider now the series which results upon multiplying both sides of this equation by n^k; k being a positive integer. This series,

$$\phi_k(n) = n^k\phi(n) = \sum_{j=-\nu}^{\infty} \frac{a_j}{n^{j-k}},$$

which again converges uniformly and absolutely for $n \geqq N$, satisfies all the conditions imposed upon ϕ; so that the integral of ϕ_k from $-\infty$ to $+\infty$ may be evaluated about the same closed path as that used for ϕ. In the Cauchy form, however

$$i^k \int \phi_k(n)e^{inx}dn = \int \phi(n)\frac{\partial^k}{\partial x^k}e^{inx}dn = \frac{\partial^k}{\partial x^k}\int \phi(n)e^{inx}dn,$$

since there can be no question as to the propriety of differentiating under the sign of integration along the closed path. Owing to the equivalence of this path with the real axis it follows that

$$\frac{\partial^k}{\partial x^k}\int_{-\infty}^{\infty}\phi(n)e^{inx}dn = \int_{-\infty}^{\infty}\phi(n)\frac{\partial^k}{\partial x^k}e^{inx}dn.$$

THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, it is permissible to differentiate the integral* (4) *repeatedly with respect to x under the sign of integration.*

In case the function

$$\Phi_1(x) = \int_{-\infty}^{\infty}\frac{\phi(n)}{in}(e^{inx} - e^{in\mu})dn$$

is considered and the above theorem applied, there results the equation

$$\Phi(x) = \frac{\partial\Phi_1}{\partial x} = \int_{-\infty}^{\infty}\phi(n)e^{inx}dn,$$

which is true for all values of x different from zero. That is $\Phi_1(x)$ is a primitive of $\Phi(x)$ which vanishes when $x = \mu$; therefore

$$\int_{\mu}^{x}dx\int_{-\infty}^{\infty}\phi(n)e^{inx}dn = \int_{-\infty}^{\infty}dn\int_{\mu}^{x}\phi(n)e^{inx}dx. \tag{16}$$

This argument breaks down if the signs of μ and x are not alike, since in this case the x-integration passes over the value $x = 0$ for which the

integral (4) can not be reduced to a closed path. In this case $\Phi(0)$ will not ordinarily possess a meaning and therefore it is impossible to assign a meaning to (16). In the special case in which $\Phi(x)$ has a meaning even when $x = 0$, this objection no longer applies and the equation (16) is still true.*

THEOREM 3. *If the conditions of Theorem 1 are satisfied, the integral (4) may be integrated with respect to x under the sign of integration, between limits of like sign.*

7. **The Fourier integral identity: the function $\Psi(\lambda, t)$.** The ideas presented above throw a rather interesting light upon the Fourier integral identity, and inasmuch as a consideration of this identity will also serve the purpose of introducing a certain function $\Psi(\lambda, t)$ which will be needed in a number of places later on, it may not be amiss to give it consideration at this time.

Consider the function

$$\Psi(\lambda, t) = i\int_{-\infty}^{\infty} \frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\, dn, \qquad (\mu \neq 0,\ t \neq \lambda). \tag{17}$$

It is Riemann integrable, if $t \neq \lambda$; but what is of more immediate consequence it may be evaluated as a Cauchy integral, as explained in section 5. Its value is thus found to be†

$$\Psi(\lambda, t) = 0, \qquad \lambda < t; \qquad \Psi(\lambda, t) = 2\pi, \qquad \lambda > t.$$

When $t = \lambda$ it is indeterminate. This uncertainty may be overcome by means of the definition

$$\Psi(\lambda, t) = 0, \qquad \lambda < t; \qquad \Psi(\lambda, t) = 2\pi, \qquad \lambda \geqq t. \tag{18}$$

Hereafter, throughout the entire paper, the notation $\Psi(\lambda, t)$ will be consistently used to refer to the function defined by (18); and will only incidentally have any relation to equation (17).

Now build up the Stieltjes integral $\int_{-\infty}^{\infty} f(\lambda)d\Psi(\lambda, t)$. Quite obviously this integral has the value $2\pi f(t)$, whatever t may be, so that it is possible to state at once the identity

$$2\pi f(t) \equiv \int_{-\infty}^{\infty} f(\lambda)d\Psi(\lambda, t). \tag{19}$$

Returning again to the consideration of (17), it is to be noted that $\partial\Psi/\partial\lambda$ may be obtained by differentiating (17) under the sign of integra-

* It is easily seen that in case $\Phi(x)$ is defined for $x = 0$, all of the integrals involved in the above discussion are convergent in the Riemann sense and there is therefore no necessity of using the Cæsaro definition.

† These values are computed on the assumption that $\mu > 0$, as it may perfectly well be.

tion, except when $\lambda = t$. Performing this differentiation, it is found that $d\Psi \equiv d\lambda \int_{-\infty}^{\infty} e^{in(t-\lambda)}dn$, for $\lambda \neq t$. When $\lambda = t$, the right-hand side of this equation is meaningless,—although there is no reason why it should not be given a meaning by definition. At present, however, this will not be done, and only formal equations will be dealt in.

Substituting the formal value of $d\Psi$ in equation (19), the result

$$2\pi f(t) = \int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} e^{in(t-\lambda)}dn \tag{20}$$

is obtained. This is the Fourier identity as ordinarily written; and the derivation given reveals its formal equivalence with the rigorous identity (19). In what follows, (19) may occasionally be termed "Fourier's Integral"; (20) being regarded as a meaningless formal equivalent, which, like derivative notation, claims recognition because of its universal familiarity.

Finally, it should be noted that $\int_a^b f(\lambda)d\Psi(\lambda, t)$ is equivalent to (19), provided $a < t \leqq b$. In case $a \geqq t$ or $t > b$ this equivalence no longer obtains.

8. **The integral**

$$G(t) = \int_a^b \chi(\lambda, t)f(\lambda)d\lambda. \tag{21}$$

In interpreting the divergent Fourier identity in the last paragraph, a very simple Stieltjes integral was found of value. The next few sections will be devoted to developing in a simple manner such of the properties of these integrals as will be of service later in assigning a meaning to a broad class of divergent integrals.

Let $f(\lambda)$ be a function of bounded variation within the closed interval (ab); and let $\chi_1(\lambda, t)$ and $\chi_2(\lambda, t)$ be two functions which are regular in λ throughout the entire interval (ab) for values of t in a certain interval (t_1, t_2) which includes (a, b). It will be assumed in the following that t takes no values outside (a, b). Furthermore, let

$$\chi_1(t, t) = \chi_2(t, t),$$

and define the function

$$\chi(\lambda, t) = \chi_1(\lambda, t) \quad (\lambda \leqq t); \qquad \chi(\lambda, t) = \chi_2(\lambda, t) \quad (\lambda \geqq t).$$

This is the function which occurs in equation (21).

It is quite obvious that (21) becomes

$$G(t) = \int_a^t \chi_1(\lambda, t)f(\lambda)d\lambda + \int_t^b \chi_2(\lambda, t)f(\lambda)d\lambda,$$

and that each of these integrals may be differentiated under the sign of integration, so that

$$\frac{\partial G}{\partial t} = \int_a^t \frac{\partial \chi_1(\lambda, t)}{\partial t} f(\lambda) d\lambda + \int_t^b \frac{\partial \chi_2(\lambda, t)}{\partial t} f(\lambda) d\lambda. \tag{22}$$

But in the interval $(a, t, \partial\chi_1/\partial t = \partial\chi/\partial t$, while in the interval $t, b)$, $\partial\chi_2/\partial t = \partial\chi/\partial t$. Thus the two integrals of (22) may be included under one sign by replacing $\partial\chi_1/\partial t$ and $\partial\chi_2/\partial t$ by $\partial\chi/\partial t$, except for the fact that the expression $\partial\chi/\partial t$ has no meaning at the point $\lambda = t$. This objection is easily overcome, however, by replacing both $\partial\chi_1/\partial t$ and $\partial\chi_2/\partial t$ by a unilateral derivative of χ; thus altering the integrand of (22) by at most a finite amount at one point only. To be explicit, the left-hand derivative will be used, so that

$$\frac{\partial G}{\partial t} = \int_a^b f(\lambda) \ \frac{\partial \chi}{\partial t} d\lambda. \tag{23}$$

As a special case, consider the function $\chi(\lambda, t)$ defined by the equation

$$\chi(\lambda, t) = \int_{-\infty}^{\infty} \phi(n) \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn, \tag{24}$$

where it is assumed that $\phi(\infty) = 0$. It is easily seen that the path of integration, which in (24) is the real axis, may be distorted into the path A of Fig. 4, provided $\phi(n)$ has no singularities infinitely near the real axis.

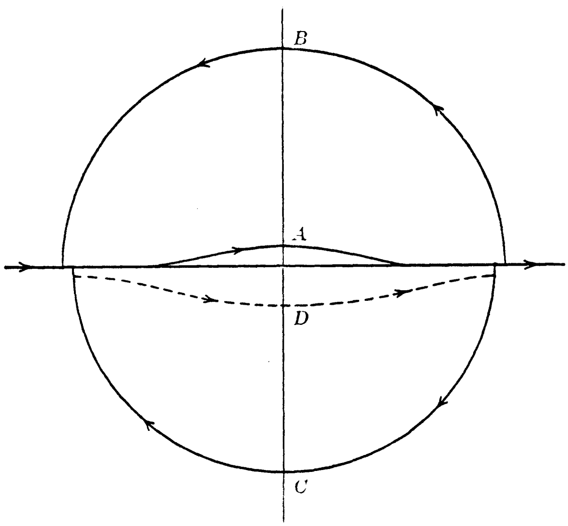

FIG. 4.

But along this path each separate term of χ is regular and may be evaluated independently. Further, each of them may be reduced to some such

path as AB or AC, according as the coefficient of n in the exponent of (24) is positive or negative.

In particular, if μ is positive, the integral

$$\int \frac{\phi(n)e^{in\mu}dn}{n} dn$$

must be taken along the path AB. It is independent of both λ and t, and may be denoted by a constant K. As for the other term,

$$\int \frac{\phi(n)e^{in(t-\lambda)}}{n} dn \tag{25}$$

it must be evaluated along the path AB when $t > \lambda$, and along the path AC when $t < \lambda$. That is, unless $t = \lambda$, χ is equal to one of the functions

$$\chi_1(\lambda, t) = -K + \int_{AB} \frac{\phi(n)e^{in(t-\lambda)}}{n} dn,$$

$$\chi_2(\lambda, t) = -K + \int_{AC} \frac{\phi(n)e^{in(t-\lambda)}}{n} dn.$$

But these functions are analytic in both λ and t, and therefore satisfy the conditions laid down at the beginning of this section. Moreover, owing to the fact that $\phi(\infty)$ has been assumed zero, it is easy to prove by an argument similar to that of section 5 that (25) may be evaluated about a closed path even when $\lambda = t$. Indeed, in this case, it does not matter which of the paths AB or AC is used, so that $\chi(t, t) = \chi_1(t, t) = \chi_2(t, t)$. This establishes the fact that χ is everywhere continuous in the interval (a, b) and therefore satisfies the conditions imposed in the discussion of equation (21).

It therefore follows immediately that (23) is satisfied by this function χ. It is also immediately seen that (23) may be written in the alternative form

$$\frac{\partial \chi}{\partial t} \int_a^b f(\lambda)\chi(\lambda, t)d\lambda = -\int_a f(\lambda)d\chi(\lambda, t), \tag{26}$$

since

$$\left| \frac{\partial \chi}{\partial t} = -\frac{\partial \chi}{\partial \lambda} \right|$$

These are the formulæ for differentiating under the sign of integration. The conditions imposed should be carefully noted. They are:

With regard to ϕ, that it is regular along the real axis, and zero at infinity, and that its essential singularities, if they exist, are of such a character that the cuts of the Riemann surface on which ϕ is analytic need not intersect the real axis; and

With regard to f, that it is a function of bounded variation in (ab).

In carrying out the proof, the tacit assumption has been made that a and b do not vary with t. This assumption is not necessary, however, provided the customary terms are added to (23) and (26).

It should also be observed that the proof is in no way dependent upon the fact that $\phi(n)$ is not a function of t, and is equally true even when ϕ varies with t. In this case, however, the alternative form (26) cannot be used, since it is no longer true that

$$\left|\frac{\partial\chi}{\partial t} = -\frac{\partial\chi}{\partial\lambda}\right|.$$

9. **The Stieltjes integral,**

$$\Phi(t) = \int_a^b f(\lambda)d\chi(\lambda, t) \tag{27}$$

Differentiation under the sign of integration. Stieltjes has shown that it is permissible to integrate an integral of the type (27) by parts, provided $f(\lambda)$ is not discontinuous at $\lambda = t$ and is a function of bounded variation.* Therefore, denoting $f(b)\chi(b, t) - f(a)\chi(a, t)$ by $F(a, b, t)$,

$$\Phi(t) = F(a, b, t) - \int_a^b \chi(\lambda, t)df(\lambda).$$

Assume for simplicitly that $f(\lambda)$ is discontinuous at one point only in the open interval† a, b; and that this point is $\lambda = \Lambda$. Denote by $2\pi\delta$ the value $f(\Lambda + 0) - f(\Lambda - 0)$. The exact value of $f(\Lambda)$ is of no consequence, since it contributes nothing to $\Phi(t)$. If it is not equal to $f(\Lambda + 0)$, a new function $f(\lambda + 0)$ may be substituted for $f(\lambda)$ in (27) without in any way altering the value of Φ. It will be assumed that this substitution has been made; but for simplicity it will not be explicitly set forth in the notation.

Consider the new function $\bar{f}(\lambda)$ which is defined as

$$\bar{f}(\lambda) = f(\lambda) - \delta\cdot\Psi(\lambda, \Lambda),$$

where Ψ is defined by (18). Then $\bar{f}(\lambda)$ is continuous, even at $\lambda = \Lambda$, and $df(\lambda)$ may be replaced by

$$\delta\cdot d\Psi(\lambda, \Lambda) + \left.\frac{d\bar{f}}{d\lambda}\right| d\lambda.$$

* The conditions stated by Stieltjes were not as broad as these; but he indicates clearly that he knew they were more restrictive than necessary.

† If f is discontinuous at an endpoint, say at a, it may be replaced by the function f_1 defined as

$$f_1 = 0 \qquad (\lambda < a); \qquad\qquad f_1 = f(\lambda) \qquad (\lambda \geqq a).$$

Then $\Phi(t)$ may be written as

$$\Phi(t) = \int_{a-\eta}^b f_1(\lambda)d\chi(\lambda, t).$$

The function f_1 is not discontinuous at the limits of integration, and has only one discontinuity In case a is a function of t, this process is still possible. The fact that two discontinuities are introduced is obviously not essential.

That is,

$$\Phi(t) = F(a, b, t) - \int_a^b \chi(\lambda, t) \cdot \frac{d\bar{f}}{d\lambda}\Big| d\lambda - 2\pi\delta \cdot \chi(\Lambda, t).$$

Differentiating, and making use of formula (23), the value of $d\Phi/dt$ is obtained in the form

$$\frac{d\Phi}{dt} = \frac{dF}{dt} - 2\pi\delta\frac{d\chi(\Lambda, t)}{dt} - \int_a^b \frac{d\bar{f}}{d\lambda}\Big| \cdot \Big|\frac{\partial\chi}{\partial t} d\lambda$$
$$- \frac{db}{dt}\frac{d\bar{f}(b)}{db}\Big| \chi(b, t) + \frac{da}{dt}\frac{d\bar{f}(a)}{da}\Big| \chi(a, t).$$

This equation is capable of very considerable simplification. In the first place,

$$\frac{d\bar{f}}{d\lambda}\Big| = \frac{df}{d\lambda}$$

except at $\lambda = \Lambda$. But since Λ is different from both b and a, the dashes can be removed from all f's which are not under the sign of integration. Furthermore,

$$2\pi\delta\,\frac{d\chi(\Lambda, t)}{dt}$$

is obviously equal to

$$\int_a^b \delta\Big|\frac{d\chi(\lambda, t)}{dt} d\Psi(\lambda, \Lambda);$$

and may be combined with the integral which follows it. But when this is done, inspection shows that the entire integral term may be rewritten in the form

$$\int_a^b \Big|\frac{\partial\chi(\lambda, t)}{\partial t} df(\lambda).$$

Collecting these results, and once more applying the formula of integration by parts it is found that

$$\frac{d\Phi}{dt} = \int_a^b f(\lambda) d\Big|\frac{\partial\chi(\lambda, t)}{dt} + \frac{db}{dt}f(b)\frac{\partial\chi(b, t)}{\partial b} - \frac{da}{dt}f(a)\frac{\partial\chi(a, t)}{\partial a}. \quad (28)$$

This is very similar to the ordinary formula of differentiation under the sign of integration. It is quite obvious that it may be extended to the case where $f(\lambda)$ has a finite number of discontinuities.

Since there is no distinction between $\partial\chi/\partial t$ and $|\,\partial\chi/\partial t$ except at the point $\lambda = t$, the value of $|\,\partial\chi/\partial t$ may be found by evaluating the formal derivative of (24),

$$\Big|\frac{\partial\chi}{\partial t} = -\int e^{in(t-\lambda)}\phi(n)dn, \quad (29)$$

about one of the closed paths AB and AC. This is immediately obvious when $\lambda \neq t$. But it has already been shown that $\chi(\lambda, t) = \chi_2(\lambda, t)$ when $t \leqq \lambda$, and that this function χ_2 is analytic in both λ and t. Hence it follows that at $\lambda = t$,

$$\left|\frac{\partial\chi}{\partial t} = \frac{\partial\chi_2}{\partial t}\right.;$$

which is the integral (29) evaluated about the path AB. Thus the propriety of differentiating (27) under both signs of integration is established and the following theorem is obtained:

THEOREM 4. *Let $f(\lambda)$ be a function of bounded variation in (a, b), which is not discontinuous at $\lambda = t$. Let $\phi(n)$ be regular along the real axis, zero at infinity, and capable of representation upon a Riemann's surface in such a way that a point traversing the entire real axis remains continually on the same sheet. Define $\chi(\lambda, t)$ as in (24), and $\Phi(t)$ as in (27). Then the derivative $d\Phi/dt$ exists, provided db/dt and da/dt do, and may be obtained by differentiation under the sign of integration in accordance with the formula* (28), *in which*

$$\frac{\partial\chi}{\partial t} = -\int e^{in(t-\lambda)}\phi(n)dn,$$

taken about AB if $t > \lambda$, otherwise about AC.

In the application to physical problems, there is no occasion to use variable limits; in consequence of which b and a will be assumed constant in what follows.

The result expressed by (28) may be extended to the case of infinite limits of integration. Assume that $f(\lambda)$ has, in the interval (a, ∞) only a finite number of discontinuities and that no point on the real axis is either a singular point or a limiting point of the singularities of $\phi(n)$. There is then a finite number b so large that in the interval (b, ∞) $f(\lambda)$ is continuous. It will be assumed that b is chosen large enough to satisfy the condition $b > t$, for all values of t which need be considered.

Then

$$\Phi_1(t) = \int_a^\infty f(\lambda)d\chi(\lambda, t) = \int_a^b f(\lambda)d\chi(\lambda, t) + \int_b^\infty f(\lambda)\frac{\partial\chi}{\partial\lambda}d\lambda.$$

The former of these integrals comes under the proof already given, and only the latter need be considered. This, however, is an ordinary Riemann integral, and may be differentiated repeatedly under the sign of integration *so long as the resultant integrals are uniformly convergent*, as they will presently be proved to be.

For all values of $\lambda > t$, and hence for all values of $\lambda \geqq b$, $\chi(\lambda, t)$ may

be found by evaluating

$$\int \phi(n) \frac{e^{in(t-\lambda)}}{n} dn$$

about the path AC, and

$$\int \phi(n) \frac{e^{in\mu}}{n} dn$$

about the path AB. The latter of these, so far as either λ or t is concerned, is a constant, which has already been denoted by K. The former, which may be called $F(\lambda, t)$ may readily be reduced to the form

$$F(\lambda, t) = -2\pi i\phi(0) + \int_{DC} \frac{\phi(n)}{n} e^{in(t-\lambda)} dn,$$

the path DC (shown in Fig. 4), being at no point removed less than a finite distance η from the real axis. Hence

$$\chi(\lambda, t) = -K - 2\pi i\phi(0) + \int_{DC} \frac{\phi(n)}{n} e^{in(t-\lambda)} dn,$$

and, whatever the values of λ and t,

$$\frac{\partial^{j+1}\chi}{\partial t^j \partial\lambda} = -i^{j+1} \int_{DC} n^j \phi(n) e^{in(t-\lambda)} dn.$$

The path DC possesses a finite length L. Along it the value of n is constantly finite and less than N; $\phi(n)$ is constantly less than a maximum value M, and

$$|e^{in(t-\lambda)}| \leqq e^{-\eta(\lambda-t)} \leqq e^{-\eta(\lambda-b)} \qquad (\lambda \geqq b \geqq t).$$

Thus

$$\left| \frac{\partial^{j+1}\chi}{\partial t^j \partial\lambda} \right| \leqq MLN^j e^{-\eta(\lambda-b)} \qquad (\lambda \geqq b \geqq t).$$

With regard to $f(\lambda)$, which has been assumed to be a function of limited variation in (a, b), it is sufficient to assume the condition

$$|f(\lambda)| < A \cdot \lambda^q \qquad (\lambda \geqq b).$$

Then it is true that, for any s greater than b,

$$\left| \int_s^\infty f(\lambda) \frac{\partial^{j+1}\chi}{\partial t^j \partial\lambda} d\lambda \right| \leqq ALMN^j e^{\eta b} \int_s^\infty \lambda^q e^{-\eta\lambda} d\lambda \leqq ALMN^j e^{-\eta(s-b)} P(s),$$

where $P(s)$ is a polynomial of degree q in s. This evidently converges to zero uniformly in t as s is increased indefinitely. *Thus the applicability of* (28) *to infinite limits is established if* $f(\lambda)$ *possesses only a finite number of discontinuities; if* $|f(\lambda)| < A \cdot \lambda^q$, $\lambda \geqq b$; *and if the singularities of* $\phi(n)$ *have no limiting points upon the real axis.*

Finally it should be noted that (28) is still true when ϕ is a function of both n and t. In this case the form of (29) is altered, since the equation as written no longer represents the result of differentiating (24) under the sign of integration.

10. **The Stieltjes integral (27): inversion of integrals.** Consider the two functions

$$\chi(\lambda, t) = i\int_{-\infty}^{\infty} \phi(n)\frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\,dn$$

and

$$\chi_1(\lambda, t) = \int_{-\infty}^{\infty} \phi(n)\frac{e^{in(t-\lambda)} - e^{in(\nu-\lambda)} - ine^{in\mu}(t-\nu)}{n^2}\,dn,$$

where $\phi(n)$ at infinity is regular, but not necessarily zero. The integrand of χ_1 has no singularities upon the real axis; and its path of integration may therefore be distorted into the path A of Fig. 4. This having been done, χ_1 may be rewritten in the form

$$\chi_1(\lambda, t) = \int_A \frac{\phi(n)}{n}\frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\,dn - \int_A \frac{\phi(n)}{n}\frac{e^{in(\nu-\lambda)} - e^{in\mu}}{n}\,dn - i(t-\nu)\int_A \frac{\phi(n)}{n}e^{in\mu}dn.$$

The third of these terms may be evaluated about the path AB, regardless of the values of t and λ. The second integral can be evaluated about either the path AB or the path AC, and is independent of t. The first term is of the type discussed in section 9, except that the $\phi(n)$ of section 9 corresponds to the $\phi(n)/n$ of this section.* Hence it is immediately obvious that the entire argument of section 9 applies equally well to the expression $\Phi_1(t)$ defined by the equation

$$\Phi_1(t) = \int_a f(\lambda)d\chi_1(\lambda, t).$$

In particular,

$$\frac{\partial\Phi_1}{\partial t} = \int_a^b f(\lambda)d\left|\frac{\partial\chi_1(\lambda, t)}{\partial t}\right.,$$

$\left|\dfrac{\partial\chi_1(\lambda, t)}{\partial t}\right.$ representing the left-hand unilateral derivative of χ.

But $\left|\dfrac{\partial\chi_1}{\partial t}\right.$ may be found by actual computation to be, equal to $\chi(\lambda, t)$, except possibly for the value $\lambda = t$; so that

$$d\left|\frac{\partial\chi_1}{\partial t}\right. = d\chi.$$

* The $\phi(n)$ of section 9 was required to have no singularities on the real axis. This condition was necessary to justify the shift from the real axis to the path A as a path of integration, and was thereafter of no consequence. This shift having already been accomplished, the fact that $\phi(n)/n$ is not necessarily regular at $n = 0$ causes no difficulty.

Therefore, if

$$\Phi(t) = \int_a^b f(\lambda) d\chi(\lambda, t),$$

it may be said that

$$\frac{\partial \Phi_1}{\partial t} = \Phi.$$

Taking account of the fact that $\Phi_1 = 0$ when $t = \nu$, it follows that

$$\Phi_1 = \int_\nu^t \Phi dt. \tag{30}$$

But, quite obviously, $\chi_1(\lambda, t)$ is the result of integrating $\chi(\lambda, t)$ with respect to t between the limits ν and t; the integration being performed *under* the sign of integration. That is (30) expresses the legitimacy of integration under both signs of integration.

11. **The Ψ-functions.** Up to the present point in this discussion no use has been made of the theory of divergent integrals as developed in the early sections. It will be the purpose of this and the remaining sections to apply this theory to the Stieltjes integral, and to the solution of differential equations. For generality, it is assumed that ϕ is a function of both n and t.

Let there be a function $\phi(n, t)$ which vanishes at infinity, and is regular over a certain range of values of t for sufficiently large values of n. Let

$$\chi(\lambda, t) = \int_{-\infty}^{\infty} \phi(n, t) \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn$$

and let

$$\Phi(t) = \int_a^b f(\lambda) d\chi(\lambda, t).$$

It has been shown that this function Φ may be differentiated under both signs of integration, even when $\lambda = t$, provided that the resulting $\left| \frac{\partial \chi}{\partial t} \right.$ is evaluated about the path AB if $t > \lambda$, and about the path AC if $t \leqq \lambda$. This is true, however, only because χ is continuous, and if $\left| \frac{\partial \chi}{\partial t} \right.$ is discontinuous $d^2\Phi/dt^2$ cannot be found in this way. It is therefore necessary to consider the magnitude of the discontinuity of $\left| \frac{\partial \chi}{\partial t} \right.$, and its effect upon the second derivative of Φ.

The conditions imposed upon ϕ justify the expansion

$$\phi(n, t) = \sum_{j=1}^{\infty} \frac{a_j(t)}{n^j};$$

which is term by term differentiable with respect to t. Making use of this form of expansion it may be said that

$$\frac{\partial^\sigma}{\partial t^\sigma}\left[\phi(n,\,t)\,\frac{e^{in(t-\lambda)}-e^{in\mu}}{n}\right] = -\frac{e^{in\mu}}{n}\frac{\partial^\sigma\phi}{\partial t^\sigma} + \sum_{j=1}^{\sigma}\sum_{k=0}^{\sigma} C_k^\sigma i^k n^{k-j-1}\frac{d^{\sigma-k}a_j}{dt^{\sigma-k}}e^{in(t-\lambda)}.$$

Some of the terms in this expansion involve powers of n higher than -2. Denote their sum by P_σ. Then

$$P_\sigma = \sum_{j=1}^{\sigma}\sum_{k=j}^{\sigma} C_k^\sigma i^k n^{j-2}\frac{d^{\sigma-k}a_{k-j+1}}{dt^{\sigma-k}}e^{in(t-\lambda)}.$$

Finally, the functions ϕ_σ and χ_σ are defined as

$$\phi_\sigma(n,\,t) = \frac{\partial^\sigma}{\partial t^\sigma}\left[\phi\,\frac{e^{in(t-\lambda)}-e^{in\mu}}{n}\right] - P_\sigma, \qquad \chi_\sigma(\lambda,\,t) = \int \phi_\sigma(n,\,t)dn,$$

the integral in the last equation being evaluated about AB if $\lambda < t$; otherwise about AC.

It is seen at once that every ϕ_σ is of degree -2 in n. Therefore every χ_σ is continuous* at $\lambda = t$, and every integral of the type $\int_a^b f(\lambda)d\chi_\sigma$ may be differentiated with respect to t under the integral signs. But by actual differentiation

$$\frac{d\phi_\sigma}{dt} = \phi_{\sigma+1} + \frac{i}{n}e^{in(t-\lambda)}\left[Q_{\sigma+1} - \frac{dQ_\sigma}{dt}\right],$$

where

$$Q_\sigma(t) = \sum_{k=1}^{\sigma} C_k^\sigma i^{k-1}\frac{d^{\sigma-k}a_k}{dt^{\sigma-k}}.$$

Hence†

$$\frac{d\chi_\sigma(\lambda,\,t)}{dt} = \chi_{\sigma+1} + \Psi(\lambda,\,t)\left[Q_{\sigma+1} - \frac{dQ_\sigma}{dt}\right]$$

and

$$\frac{\partial\Phi}{\partial t} = \int_a^b f(\lambda)d\,\frac{\partial\chi}{\partial t} = \int_a^b f(\lambda)d(\chi_1 + a_1\Psi) = 2\pi a_1(t)f(t) + \int_a^b f(\lambda)d\chi_1(\lambda,\,t).$$

Having once obtained this formula, it is easy to establish by induction the following theorem:

THEOREM 5. *Let $\phi(n,\,t)$ be a function which, for a certain range of values of t to which attention is confined, and for n real or sufficiently large, is regular in both t and n, and which vanishes at $n = \infty$. Then the Stieltjes Integral $\Phi(t)$ possesses a σ'th derivative provided $f(t)$ possesses a $\sigma - 1$'th*

* See footnote on page 33.

† The identification of $-\int \frac{e^{in(t-\lambda)}}{in}dn$ with $\Psi(\lambda,\,t)$ is established by direct evaluation.

derivative; and this derivative may be obtained by differentiating the Stieltjes integral Φ *under all signs of integration, according to the formula*

$$\frac{\partial^\sigma \Phi}{\partial t^\sigma} = 2\pi\left[\frac{\partial^{\sigma-1}(a_1 f)}{\partial t^{\sigma-1}} + \sum_{j=0}^{\sigma-2}\frac{\partial^j}{\partial t^j}\left(fQ_{\sigma-j} - f\frac{\partial Q_{\sigma-j-1}}{\partial t}\right) + \int_a^b f(\lambda)d\chi_\sigma(\lambda, t)\right]. \tag{31}$$

Equation (31) gives the true value of the derivatives of Φ, by introducing into the formal derivative certain auxiliary terms. For purposes of comparison it is desirable to write down the result of differentiating $\Phi(t)$ formally under all signs of integration. The result is

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)d\chi^{(\sigma)}, \tag{32}$$

where

$$\chi^{(\sigma)}(\lambda, t) = \int_{-\infty}^{\infty}[\phi_\sigma(n, t) + P_\sigma(n, t)]dn. \tag{33}$$

It has been shown that this *formal* result for $\chi^{(\sigma)}$ may be evaluated about one of the paths AB or AC, *so long as* $t \neq \lambda$. What is not known is *that when the result is substituted in* (32) *it results in* $\partial^\sigma\Phi/\partial t^\sigma$; and indeed, so long as $d\chi^{(\sigma)}$ does not have a meaning at $\lambda = t$, the statement in italics is absurd. Ignoring this difficulty for the moment, consider the actual value of $\chi^{(\sigma)}$ for $t \neq \lambda$. Only those terms of P_σ for which the exponent of n is -1 contribute anything to the result. Hence, separating out these terms it is seen that

$$\int_{-\infty}^{\infty} P_\sigma(n, t)dn = \Psi(\lambda, t)Q_\sigma(t),$$

and therefore

$$d\chi^{(\sigma)}(\lambda, t) = d\chi_\sigma(\lambda, t) + Q_\sigma(t)d\Psi(\lambda, t) = d\chi_\sigma(\lambda, t) \qquad (\lambda \neq t). \tag{34}$$

This is exactly the function with respect to which the integral term of (31) is integrated for all values of λ.

The first observation to which this equality leads is concerned with the result of a substitution of (34) in (31). Suppose (33) to be evaluated about AC when $\lambda = t$; that is, suppose the same convention regarding the path of integration is made in (33) as in (31). Then (34), in its first form, is true even when $\lambda = t$, and therefore *it is not true that*

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)d\left|\frac{\partial^\sigma \chi}{\partial t^\sigma}\right.,$$

in the general case. In fact, the deficiency of this equation is exactly the difference of the group of terms which do not occur under the sign of integration in (31), *and the term* $2\pi f(t)Q_\sigma(t)$.

The second observation concerns those terms of P_σ which contribute nothing to the value of (33) when $\lambda \neq t$. *Formally* the corresponding terms of (33) are the successive derivatives of the equation (17), multiplied by a factor independent of n. *Formally*, therefore, (32) becomes

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)\left[d\chi_\sigma + \sum_{j=1}^{\sigma}\sum_{k=j}^{\sigma} C_k^\sigma \frac{d^{\sigma-k}a_{k-j+1}}{dt^{\sigma-k}} i^{k-j} d\Psi^{(j-1)}\right].$$

This result is significant, for if the definitions

$$\int_a^b f(\lambda) d\Psi^{(\sigma)}(\lambda, t) = 2\pi f^{(\sigma)}(t), \tag{35}$$

which are suggested by the *formal* differentiation of (19), are made for *all* values of σ, it follows that (32) is a true equation.*

The definitions (35), however, are sufficient to give a meaning to an integral of the type (32) regardless of how it may have been derived—whether by differentiation or otherwise—and are therefore of very general importance. In fact, it requires but a moment's reflection to observe the truth of the following statements:

DEFINITION 1. *Let $\Psi^{(\sigma)}(\lambda, t)$ represent the divergent integral*

$$i^{\sigma+1}\int_{-\infty}^{\infty} n^{\sigma-1} e^{in(t-\lambda)} dn.$$

Then the identity

$$2\pi f^{(\sigma)}(t) \equiv \int_a^b f(\lambda) d\Psi^{(\sigma)}(\lambda, t)$$

defines the integral in its right-hand member, provided $a < t < b$.

DEFINITION 2. *Let $\phi(n)$ be a function which, for a certain range of values of t to which attention is confined, and for real values of n, is regular in both n and t; and which may be expanded in a series*

$$\phi(n, t) = \sum_{j=-v}^{\infty} \frac{a_j(t)}{n^j},$$

convergent for sufficiently large values of n. Also let $\chi(\lambda, t)$ be defined by the equation

$$\chi(\lambda, t) = \int_{-\infty}^{\infty} \phi(n, t) e^{in(t-\lambda)} d\lambda.$$

* This is shown, of course, by establishing that the equation preceding (35) is equivalent to (31). The algebraic manipulation involved is somewhat simplified by the use of the identities:

$$(a)\ C_k^\sigma = C_k^{\sigma-1} + C_{k-1}^{\sigma-1}, \qquad (b)\ C_k^\sigma = \sum_{j=1}^{\sigma-k} C_{k-1}^{\sigma-1}, \qquad (c)\ C_{\alpha+\beta+1}^{\sigma+1} = \sum_{j=\beta}^{\sigma-\alpha} C_\beta^j C_\alpha^{\sigma-j}.$$

Those who are not familiar with these identities may establish (a) by direct addition; (b) by repeated application of (a); and (c) by induction, assuming it to hold for all values of σ and β when $\alpha = \alpha$ and showing that this involves its validity for $\alpha = \alpha + 1$. In this process $C_\alpha^{\sigma-j}$ is replaced by the summation (b); and it is noted that when $\alpha = 0$, (c) reduces to (b).

Then the identity

$$\int_a^b f(\lambda)d\chi(\lambda, t) \equiv -2\pi \sum_{j=-v}^{0} i^j a_j(t) f^{(1-j)}(t) + \int_a^b f(\lambda)d\chi(\lambda + 0, t)$$

defines the integral in its left-hand member, provided $a < t < b$.

THEOREM 6. *The definitions* 1 *and* 2 *are consistent with the operations of addition, differentiation, integration, and multiplication by a constant when these operations are performed under the sign of integration. Furthermore, if by any of these processes an integral is derived which may be evaluated as an iterated improper Riemann integral, the value of this Riemann integral is consistent with the value obtained by performing the same operations upon*

$$\Phi(t) = \int_a^b f(\lambda)d\chi(\lambda, t).$$

Not all of the details of proof of this general result are to be found in the preceding sections. It may be well, therefore, to rapidly indicate them in this place.

In the matter of addition and multiplication by a constant there are no difficulties which need be considered.

As for differentiation, the difficulty is only slightly greater. For if

$$\Phi(t) = \int_a^b f(\lambda)d\chi(\lambda, t)$$

is a true equation when interpreted as above explained, it is possible to separate out all the terms of $\phi(n, t)$ of degree greater than -2 in n. These having been removed from the integral, the remaining terms of the integrand satisfy the conditions of the proof of Theorem 5, and repeated differentiation of Φ is permissible. The equivalence of the result thus derived with what would have been obtained had these terms not been removed is easily established.

The case of integration is made to depend upon that of differentiation. For if $\phi(n, t)$ satisfies the conditions of definition 2, the same is true of the function $\phi_1(n, t)$ defined by the equation

$$\phi_1(n, t) = e^{-int} \int_\mu^t e^{int}\phi(n, t)dn;$$

the path of integration being the real axis. Then the function

$$\Phi_1(t) = \int_a^b f(\lambda)d\chi_1(\lambda, t),$$

where

$$\chi_1(\lambda, t) = \int_{-\infty}^{\infty} \phi_1(n, t)e^{in(t-\lambda)}dn,$$

may be differentiated, the result of course, being the $\Phi(t)$ used above. Hence

$$\Phi_1(t) = \int_\mu^t \Phi(t)dt + C;$$

and it is easily seen that C must be zero, since $\Phi_1(\mu) = 0$. Since χ_1 is the result obtained by formally integrating χ from μ to t under the sign of integration, the desired property of integrability is established.

That the second sentence of the theorem is true may be verified from the fact that, if the conditions on $\phi(n)$ are satisfied, $d\chi$ cannot be Riemann integrable unless $\phi(\infty)$ is zero to the second order. But if by the use of the operations of addition, multiplication, differentiation and integration a function ϕ is built up which satisfies this condition, its expansion in descending powers of n will involve no terms which require the use of definition 2. The result will be found by altering the path of integration, a method which in this case is consistent with both Riemann integration and integration by the method here explained. Hence the last sentence of the theorem is true.

Theorem 6 is the principal result of this paper. It gives a meaning to many expressions which, so far as the writer is aware, have never been interpreted before. That many of these expressions could be manipulated successfully, provided the form in which they were left by the last transformation could be interpreted by ordinary means has been generally recognized by applied mathematicians; but their use was always considered as belonging to the class of questionable operations, and whenever it was found possible to do so, the results obtained were subjected to independent checks. It now appears, not only that reliance can be placed in the final results, even when they are not interpretable by the customary means, but also that each separate step of the transformations is rigorously justifiable.

It is not within the scope of this paper to enumerate the uses which may be made of the above theorems; but it may not be amiss to give one or two examples of their application to the solution of differential equations. These examples are drawn from the paper on "The Solution of Circuit Problems" to which reference has been made in the introduction, where they are treated in detail, although, of course, with a minimum of theoretical mathematics. They will be given here as briefly as possible, the reader being referred to the original article for technical elaboration.

The first of these problems, which is given in section 12, deals with an extremely simple electrical circuit, and is chosen for its didactic value only. The second, which is much more difficult, deals with the solution

of the set of partial differential equations controlling the propagation of electricity along a pair of parallel conductors. The third problem is equivalent to the solution of a set of linear differential equations of arbitrary degree. It is discussed in section 14.

12. **Example of the foregoing theory.** The flow of electricity in a circuit containing a resistance and capacity in series is in accordance with the equation

$$RI + K\int_{-\infty}^{t} Idt = E(t), \tag{36}$$

where I is the current flowing, E is the applied electromotive force, and R and K are the resistance and stiffness (reciprocal of capacity) of the circuit.

If E is a periodic function, e^{int}, then I is periodic of the same period, and the solution of the equation is*

$$I = \frac{in}{Rin + K}E(t).$$

Any form of E may be expressed by the equation

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dne^{int}\int_{-\infty}^{\infty} d\lambda E(\lambda)e^{-in\lambda},$$

which represents, when physically interpreted, a summation of periodic terms, each multiplied by an amplitude factor

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} E(\lambda)e^{-in\lambda}d\lambda.$$

It is natural, therefore, to expect a solution for I in the form of a summation of these same terms, modified by the factor $in/(Rin + K)$. The tentative result therefore is

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} dn\int_{-\infty}^{\infty} d\lambda E(\lambda)e^{in(t-\lambda)}\frac{in}{Rin + K}.$$

That this is the true solution is found by substitution in the equation to be solved.

The integration with respect to n is carried out along the path AB if $\lambda < t$; otherwise along AC. Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{in(t-\lambda)}\frac{in}{Rin + K}dn = -\frac{K}{R^2}e^{-\frac{K}{R}(t-\lambda)},$$

* The evaluation of $\int_{-\infty}^{t} Idt$ requires the use of the Cæsaro value of an improper integral.

if $\lambda < t$. Otherwise it is zero. The result for I is therefore apparently

$$I = -\frac{K}{R^2}\int_{-\infty}^{t} E(\lambda)e^{-\frac{K}{R}(t-\lambda)}d\lambda.$$

But that this result is deficient follows from the fact that there is a term of zero degree in the expansion of $in/(Rin + K)$ in descending powers of n. To this term corresponds the supplementary I term E/R, so that finally

$$I = \frac{1}{R}E(t) - \frac{K}{R^2}\int_{-\infty}^{t} E(\lambda)e^{-\frac{K}{R}(t-\lambda)}d\lambda.$$

For instance, if $E(t) = 0$ for $t < 0$ and $E(t) = 1$ for $t > 0$, then I is zero unless $t > 0$, in which case

$$I = \frac{1}{R} - \frac{K}{R^2}\int_{0}^{t} e^{-\frac{K}{R}(t-\lambda)}d\lambda = \frac{1}{R}e^{-\frac{K}{R}t}.$$

That this result is correct is capable of direct verification by substitution in the original equation. *There is no need for this,* however, except as a check upon the computation, *for the method has been thoroughly established.*

This simple problem may be solved in easier ways; but it serves the purpose in this place of showing how the definitions which have been given above overcome the deficiencies in an otherwise powerful method. A more formidable example occurs in the next section.

13. **Application to the solution of differential equations: the telegraph equation.** As stated in the introduction, the writer conceived the ideas which have formed the material for this paper from a consideration of the problem of the telegraph equation. This problem is here presented as briefly as possible, and serves to indicate the application of the Fourier integral to the solution of partial differential equations.

The propagation of electricity along a pair of parallel conducting wires takes place in accordance with a system of differential equations, which, if units are properly chosen, reduce to

$$-\frac{\partial E}{\partial x} = \frac{\partial I}{\partial t} + I, \qquad -\frac{\partial I}{\partial x} = \frac{\partial E}{\partial t} + kE. \tag{37}$$

If both I and E are assumed to be periodic in t, a solution is readily found in the form

$$E(x, t) = e^{int}[Q_1e^{imx} + Q_2e^{-imx}],$$

$$I(x, t) = -e^{int}\frac{im}{in+1}[Q_1e^{imx} - Q_2e^{-imx}],$$

$$im = \sqrt{(in+1)(in+k)}.$$

On the other hand, if E is not periodic, it can, for a given value of x, say $x = 0$, be expanded in a Fourier integral

$$E(0, t) = \int_{-\infty}^{\infty} e^{int} F(n) dn,$$

which represents a summation of such terms as the periodic one above. It is not unnatural, therefore, to expect a solution for E in the form

$$E(x, t) = \int_{-\infty}^{\infty} [Q_1(n)e^{imx} + Q_2(n)e^{-imx}]e^{int} dn.$$

If this is postulated, it is seen that

$$Q_1 + Q_2 = F = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)e^{-in} \, d\lambda,$$

wherefore, introducing the notation

$$\frac{q_i(n)}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)e^{-in\lambda} d\lambda = Q_i(n),$$

it follows that

$$E(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} E(0, \lambda)e^{in(t-\lambda)}[q_1(n)e^{imx} + q_2(n)e^{-imx}]d\lambda.$$

Inverting the order of integration, and writing

$$\chi(\lambda, t) = i\int_{-\infty}^{\infty} \frac{e^{in(t-\lambda)} - e^{in(t-\mu)}}{n} [q_1(n)e^{imx} + q_2(n)e^{-imx}]dn \qquad (38)$$

this takes the form

$$E(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda) d\chi(\lambda, t).$$

It may be true that $q_1(n)e^{imx}$ and $q_2(n)e^{-imx}$ are essentially singular at infinity, owing to the exponential factors; but this complication will cause little difficulty. As to the location of the other singularities nothing can be affirmed in general from purely mathematical considerations. They depend upon the boundary conditions of the problem, and may conceivably lie either upon, above or below the real axis. Physical considerations show, however, that in all actual cases they must lie in the upper half plane, except when the problem has been idealized to such an extent as to put some of them upon the real axis. Even in this latter case, the same physical considerations warrant the use of a path of integration sufficiently far below the real axis to avoid these singularities.

For the purpose of this argument it will be assumed that all points of the real axis are regular points, and that the cuts are fortuitously dis-

tributed, as required throughout the preceding sections. It is then possible to vary the path of integration to such an extent that it will avoid the origin, and thereafter to split (38) into four portions involving one exponential each. Consider only the first of these,

$$\chi_1(\lambda, t) = i\int_A e^{in(t-\lambda)+imx}\frac{q_1(n)}{n}dn, \tag{39}$$

which is representative of all. It is seen that, at infinity, im is approximately equal to* $-in + [(k+1)/2]$, so that the quantity $im - in$ is finite and regular there. Indeed, it is regular at all points of the complex plane, excepting the two winding points $n = i$ and $n = ki$. This having been established, (39) may be thrown into the form

$$\chi_1(\lambda, t) = i\int_A e^{in(t-\lambda-x)}\phi_1(n)dn,$$

where

$$\phi_1(n) = \frac{q_1(n)}{n}e^{i(m-n)x}.$$

This function ϕ_1 satisfies all the conditions which have been imposed upon the function ϕ throughout this paper as regards singularities. It follows that $\int_{-\infty}^{\infty} E(0, \lambda)d\chi_1$, and therefore $\int_{-\infty}^{\infty} E(0, \lambda)d\chi$, may be subjected to the processes of repeated differentiation and integration with respect to either x or t.

Suppose then that the integrals

$$\begin{aligned} E(x, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)d\lambda\int_{-\infty}^{\infty} e^{in(t-\lambda)}[q_1(n)e^{imx} + q_2(n)e^{-imx}]dn \\ I(x, t) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)d\lambda\int_{-\infty}^{\infty}\frac{im}{in+1}e^{in(t-\lambda)}[q_1(n)e^{imx} - q_2(n)e^{-imx}]dn \end{aligned} \tag{40}$$

are substituted in the equations (37). Upon performing all the operations under the signs of integration, and interpreting the results according to the principles of section 11, it is seen that (37) is satisfied.

The first thought which is suggested by a glance at (40) is that a solution which appears in such a complicated form is of very little use. But it must be observed that a method of evaluation has been obtained which does not require an improper integration with respect to n—and when the complicated n—integral is removed from consideration and the expressions are written in the form $\int_{-\infty}^{\infty} E(0, \lambda)d\chi(\lambda, t)$, they *appear* extremely simple. Whatever difficulty may remain is due to the complicated

* The sign has been arbitrarily chosen to agree with the original paper.

nature of the boundary conditions, which will manifest itself in complicated values for q_1 and q_2.

The actual form which these functions take in dealing with the submarine cable, and the difficulties of numerical evaluation are discussed in the paper on "The Solution of Circuit Problems," to which attention has been invited above, and to which the reader who is interested in following the matter further is referred.

14. **Application to the solution of differential equations: the generalized Heaviside expansion.** As a further example, consider the set of differential equations,

$$
\begin{aligned}
a_{11}\left(\frac{\partial}{\partial t}\right)I_1 + a_{12}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{1s}\left(\frac{\partial}{\partial t}\right)I_s &= f(t), \\
a_{21}\left(\frac{\partial}{\partial t}\right)I_1 + a_{22}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{2s}\left(\frac{\partial}{\partial t}\right)I_s &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdot, \\
a_{s1}\left(\frac{\partial}{\partial t}\right)I_1 + a_{s2}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{ss}\left(\frac{\partial}{\partial t}\right)I_s &= 0,
\end{aligned}
\tag{41}
$$

in which a_{11}, a_{12}, $\cdots$, a_{ss} are linear differential operators of any order. Denote by Δ the determinant

$$
\Delta = \begin{vmatrix}
a_{11}(in) & a_{12}(in) & \cdots & a_{1s}(in) \\
a_{21}(in) & a_{22}(in) & \cdots & a_{2s}(in) \\
\cdot & \cdot & \cdot & \cdot \\
a_{s1}(in) & a_{s2}(in) & \cdots & a_{ss}(in)
\end{vmatrix}
$$

and by M_{ij} the minor of a_{ij}, and build up the expressions

$$
I_j = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} \frac{M_{1j}(in)}{\Delta(in)} e^{in(t-\lambda)}dn. \tag{42}
$$

If Δ has no roots for real values of n, these expressions may be repeatedly differentiated, and hence may be substituted in the set of equations (41). This being done, there result the expressions

$$
Q_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} \frac{1}{\Delta(in)} \sum_{j=1}^{s} a_{ij}(in)M_{1j}(in)e^{in(t-\lambda)}dn.
$$

But $\sum_{j=1}^{s} a_{ij}(in)M_{1j}(in)$ is equal to zero or Δ, according as i is or is not different from unity. Hence it follows that $Q_i \equiv 0$, unless $i = 1$, in which case it reduces to $f(t)$. That is (42) is a solution of the system of differential equations (41).

The functions M and Δ are polynomials in terms of in. Suppose

that M/Δ is divided out until a remainder N is obtained which is of degree lower than that of Δ; and that N/Δ is then expanded, as it may be, in a set of partial fractions. Formally the result will be*

$$\frac{M}{\Delta} = \sum_{k=0}^{K} p_k(in)^k + \sum_{k=1}^{K'} \frac{q_k}{in - in_k},$$

where K' is the degree of Δ in in. Substituting this result in (42) and evaluating the terms of the second summation by the method of section 5, and the terms of the first summation by the method of section 11, it is found that

$$I_j = \Sigma p_k \frac{\partial^k}{\partial t^k} f(t) + \Sigma q_k^+ \int_a^t f(\lambda) e^{in_k(t-\lambda)} d\lambda + \Sigma q_k^- \int_t^b f(\lambda) e^{in_k(t-\lambda)} d\lambda. \quad (43)$$

Here $q_k{}^+$ denotes one of the terms in which n_k lies in the upper half of the complex plane, and $q_k{}^-$ one of the terms in which n_k lies in the lower half.

In case every p_k and $q_k{}^-$ is zero, and $f(\lambda) = \Psi(\lambda)/2\pi$ this reduces to Heaviside's formula†

$$I = \sum_{k=0}^{K'} \frac{q_k}{in} (1 - e^{int}).$$

From a mathematical standpoint, the formula (43) is principally important, in that it represents a particular solution of a set of linear differential equations, written in a very explicit form. It is hardly necessary to observe that a great deal could be said about boundary conditions, and complementary solutions and the like. But these matters are of sufficient importance to justify a title of their own, and like certain others which have been mentioned in the course of the argument, must be passed by for the present.

15. **Resumé.** The paper consists of three parts, the content of which is as follows:

(*a*) A discussion of the Cæsaro limit of a class of divergent integrals leads to the conclusion that it may be found by evaluating a Cauchy integral about well-defined paths.

(*b*) It is found that an important class of Stieltjes integrals may be differentiated and integrated under the sign of integration, provided that, when differentiation is performed, left-hand unilateral derivatives are used.

* Provided the roots of Δ are all different. If any of them is repeated the form of the expansion is different.

† This formula, the importance of which in circuit theory is sufficiently attested by the fact that it possesses a name, was stated by Oliver Heaviside without proof.

(*c*) The results of the two preceding parts of the paper are brought to bear upon a class of double integrals which, in any ordinary sense, are without meaning. With the aid of a fortuitously chosen set of definitions of $\int_a^b f(\lambda)d\Psi^{(\sigma)}(\lambda)$, it is shown how these can be assigned meanings consistent with those operations to which they are most likely to be subjected in the solution of differential equations. Several examples illustrate the use of the theory.

The purpose underlying the choice and development of the material is not to exhaust its possibilities along any one line; but rather to indicate a general line of research which, it is hoped, will be found of sufficient practical value to justify its more extensive development.

RESEARCH LABORATORY OF THE AMERICAN TELEPHONE AND TELEGRAPH COMPANY AND THE WESTERN ELECTRIC COMPANY, INC.,
November, 1918.